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## LETTER TO THE EDITOR

# Static and dynamic properties for growth models

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**Abstract.** How to measure static properties in dynamic models? We propose a basic approach that should help keep the statics and the dynamics of growth models clearly separated. Our formalism is used to elaborate a large cell renormalisation group scheme which gives very good results for the determination of the fractal dimension of Eden and DLA models. We also propose a dynamic scaling theory for growth models which we verify for both Eden and DLA models.

Models for growth phenomena have received an enormous amount of attention in recent years owing to their relevance to a variety of physical and biological phenomena and to their intrinsic theoretical interest (for a review see [1]).

The most interesting characteristic for these models is the dependence of the root-mean-square radius of gyration  $\xi = \langle R_N^2 \rangle^{1/2}$  on the size  $N$  of the aggregate. For structures with scale invariance  $\xi$  grows as  $\xi \sim N^{1/d_f}$ , where  $d_f$  is the fractal dimension of the aggregate [2]. Attempts to calculate the fractal dimension via a renormalisation group scheme have given poor results [3-5]. For the diffusion-limited aggregation (DLA) model [6] and the Eden model [7] the renormalisation procedure is based on a cell-to-site mapping of spanning configurations, limiting the growth inside the cell. For small cells the method gives reasonable results, but for large cells the agreement with the accepted value for  $d_f$  becomes worse, resulting in a very slow convergence even for the simple case of the Eden model [5]. This is in contrast to static problems such as percolation or lattice animals where the convergence is very good. Other variations of this method also produced unsatisfactory results [8].

We believe that a proper way to treat the problem is to take into account the intrinsically dynamic nature of the growth process. In the DLA model the internal core of the aggregate in the long-time regime is stationary, while the outer part, where the growth process mostly occurs, changes very drastically with time. Therefore by limiting the growth inside the boundaries of the cell one cannot reproduce correctly the stationary configurations of the internal core.

The distinction between static and dynamic features of the aggregates produced by growth processes is the main issue of this letter. We will introduce a formalism to define a measure for each given configuration in a box. This allows us to calculate all static and dynamic properties inside the box. We will show on the DLA and Eden models that a simple renormalisation group applied to the equilibrium configurations

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gives good estimates for the fractal dimension. We also present a dynamic scaling theory for growth models, which compares very favourably with our measured data for the DLA and Eden models.

Let us start by defining every quantity of interest in terms of the growth site probabilities  $\{p_i\}_{i \in \Gamma}$ , namely the set of probabilities  $p_i$  that site  $i$  becomes part of the aggregate for the generic growth model. In most cases the growth can only occur on the perimeter  $\Gamma$  of the aggregate. We will also indicate by  $t$  the number of time steps to grow the aggregate, which coincides with the total number of particles in the aggregate  $N$ .

The probability  $p(C)$  that a given cluster configuration of  $N$  particles occurs in a growth model is given by

$$p(C) = \sum p_1 p_2 \cdots p_N = \sum p_{1, \dots, N} \quad (1)$$

where  $p_{1, \dots, N}$  is the probability for a given sequence of  $N$  steps and is given by the product of the probabilities  $p_i$  of adding a particle at each step. The sum is over all sequences of  $N$  steps leading to the configuration  $C$ .

Since the aggregate evolves in the course of time, one might ask how to calculate static properties. To answer this question we will consider an imaginary box of length  $L$ . Let an aggregate grow from a seed at the centre of the box *regardless of the presence of the box*. After a large number of time steps the aggregate will start to grow outside the box. Consider now the cluster configuration  $C$  inside the box. The probability  $p(C, L, t)$  that such a configuration is realised at time  $t$  is given by

$$p(C, L, t) = \sum p_{1, \dots, t} \quad (2)$$

where  $p_{1, \dots, t}$  is the probability that a given sequence of  $t$  steps leads to the configuration  $C$  *in the box* and the sum is over all possible such sequences.

The equilibrium distribution is given by

$$p(C, L) = \lim_{t \rightarrow \infty} p(C, L, t). \quad (3)$$

In this way we can calculate any static quantity in the box  $L$  and take the thermodynamic limit  $L \rightarrow \infty$ .

As an example, the mass or number of particles  $M(L, t)$  in the box  $L$  at time  $t$  is given by

$$M(L, t) = \sum p(C, L, t) M(C) \quad (4)$$

where  $M(C)$  is the mass contained in the box in the configuration  $C$ . At equilibrium the mass in the box  $M_{\text{eq}}(L)$  is simply

$$M_{\text{eq}}(L) = \lim_{t \rightarrow \infty} M(L, t) \quad (5)$$

from which an effective fractal dimension  $d_f(L)$  relative to the box of size  $L$  can be defined as

$$d_f(L) = \frac{\ln M_{\text{eq}}(L)}{\ln L} \quad (6)$$

and then the fractal dimension of the aggregate  $d_f$  is given by

$$d_f = \lim_{L \rightarrow \infty} d_f(L). \quad (7)$$

In this way we have been able to construct a measure (2) for each configuration inside the box. For large  $t$  the growth occurs mainly outside the box without affecting the configuration in the box. Thus the measure becomes independent of  $t$ .

Note that the procedure (5)-(7) to calculate the fractal dimension can also be recast in the language of a large-cell renormalisation group [9] as elaborated in [10]. In this approach if  $m_0$  is the mass of the single particle in the aggregate, the renormalisation procedure maps the cell into a single site of mass

$$m'_0 = \lambda m_0 \quad (8)$$

where  $\lambda = M_{\text{eq}}(L)/m_0$  is the average number of particles in the cell after reaching equilibrium. Equation (8) gives a recursion relation with eigenvalue represented by  $\lambda$  and with a 'critical' exponent or fractal dimension given by (6) for  $m_0 = 1$ . The expression in (6) gives therefore the approximate value relative to a cell of size  $L$ , which approaches its true value in the infinite-cell limit.

We have measured for the square lattice the mass  $M(L, t)$  contained in a cell of size  $L$  at time  $t$  (4) by generating a large number of clusters of  $t$  particles and averaging over the configurations which span the cell in the four directions. For generating the clusters both the Eden and DLA algorithms have been used. (The Eden cluster is grown by starting from a seed and adding a particle at each time step on a randomly selected perimeter site, each perimeter site having *a priori* the same probability to be chosen. The DLA cluster is grown by starting from a seed, and letting random-walking particles, launched one at a time from far away, accrete, sticking irreversibly as soon as they contact the already formed structure.) The program we used stopped the growth of each cluster at fixed intervals (for example, every 150 particles) and measured the number of particles included in a series of concentric cells superimposed on the cluster.

In figures 1(a) and 2(a) we have plotted the 'effective' time-dependent fractal dimension

$$\bar{d}_f(L, t) = \frac{\ln M(L, t)}{\ln L} \quad (9)$$

from which  $d_f(L) = \lim_{t \rightarrow \infty} \bar{d}_f(L, t)$  can be obtained.

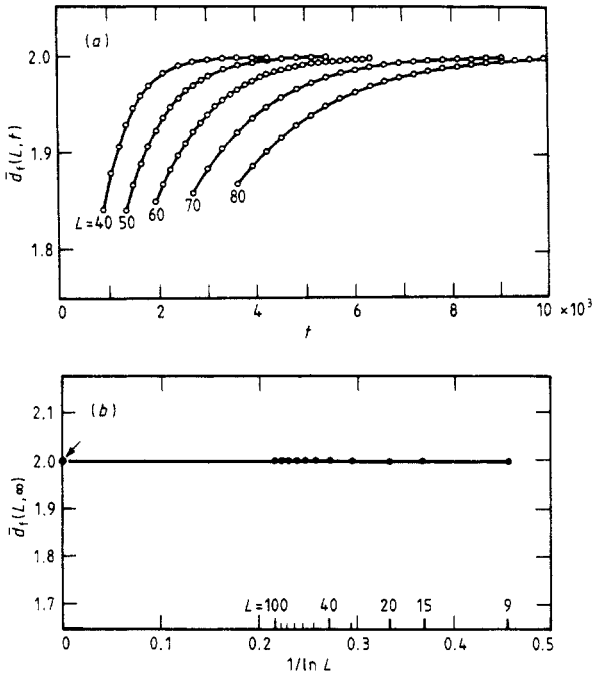
Finally extrapolating  $d_f(L)$  for  $L \rightarrow \infty$  we obtain the fractal dimension  $d_f$ . Figure 1(b) shows the results for the Eden model: one extrapolates to the right value  $d_f = 2$  from a line  $d_f(1/\ln L)$  constant in  $1/\ln L$ , showing the great advantage of this method over previous renormalisation groups, where the convergence was very slow.

A good convergence is also found for DLA (see figure 2(b)), where  $d_f(L)$  converges to  $d_f = 1.696$ , in excellent agreement with the best estimates for the fractal dimension of DLA clusters.

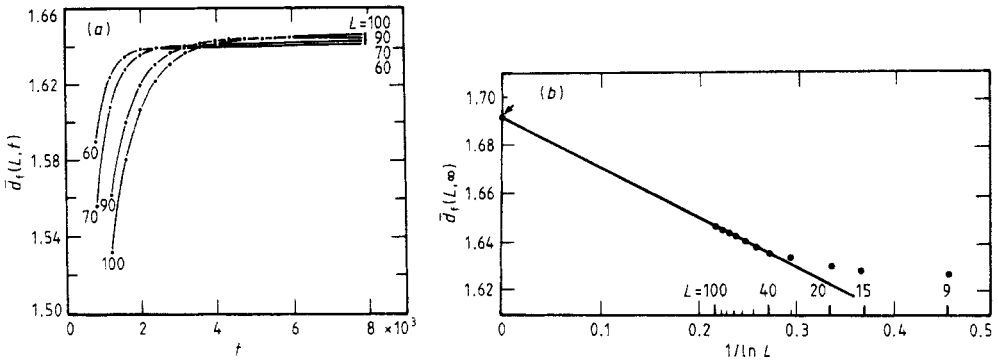
To find the dynamic behaviour using a scaling approach we can write

$$M(L, t) = M(L, \infty) f(t/\tau). \quad (10)$$

Here  $M(L, \infty) \sim L^d$ .  $\tau$  is a characteristic time above which the mass inside the box does not change any more; we assume that  $\tau$  scales as the length of the box to some unknown exponent  $z$ , i.e.  $\tau \sim L^z$ . We have  $f(x) \rightarrow \text{constant}$  for  $x \gg 1$  and  $f(x) \sim x^{d/z}$  for  $x \ll 1$ , since the mass inside the box must be independent of  $L$  for large enough  $L$  and small enough  $t$ . Moreover for  $x \ll 1$  we also have  $M(L, t) = t$ , since all the mass is inside the box. Thus we can conclude that  $z = d_f$ .



**Figure 1.** Eden model. (a)  $\bar{d}_r(L, t)$  against  $t$ ; for each  $L$  the curve reaches the limit value  $\bar{d}_r(L, \infty) = 2$ . (b)  $\bar{d}_r(L, \infty)$  against  $1/\ln L$ ; the asymptotic intercept, marked by an arrow on the graph, represents the fractal dimension of the model  $d_f = 2$ .



**Figure 2.** DLA model. (a)  $\bar{d}_r(L, t)$  against  $t$ ; for each  $L$  the curve goes to a different limit. (b)  $\bar{d}_r(L, \infty)$  against  $1/\ln L$ ; the asymptotic intercept, marked by an arrow on the graph, represents the fractal dimension of the model  $d_f = 1.691 \pm 0.010$ . The points shown have a statistics ranging from 500 to 1000 trials. Note that to obtain the points up to  $L = 100$  we had to produce clusters of the order of 8000 particles.

Using the data of figure 1 we can measure the dynamic exponent  $z$  and check our prediction. In fact a plot of  $M(L, t)/M(L, \infty)$  as a function of  $t$  gives us  $\tau(L)$  as the time value in which every curve assumes the same arbitrary reference value. A plot of  $\ln \tau(L)$  against  $\ln L$  then gives  $z$ .

Figure 3(a) illustrates our results for the Eden model, and shows clearly that we get  $z_E = 2$ . Figure 4 is related to our measures on the DLA model, from which we get

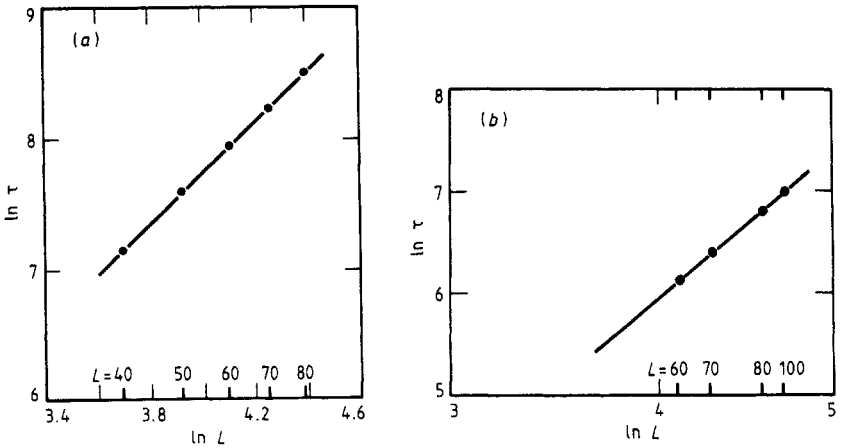


Figure 3. Plot of  $\ln \tau$  against  $\ln L$ . (a) Eden model giving slope  $z_E = 2$ . (b) DLA model giving slope  $z_{DLA} = 1.694 \pm 0.010$ .

$z_{DLA} = 1.694 \pm 0.010$ , which compares very well with the value we got for  $d_f = 1.691 \pm 0.010$ .

Finally figure 4 shows the data collapse for the Eden and DLA models obtained by plotting  $M(L, t)/M(L, \infty)$  against  $t/\tau$ .

In conclusion we have developed a formalism which allows us to calculate correctly the static properties of growth phenomena and to study their dynamic behaviour. We have applied this formalism to define a renormalisation group procedure based on a cell-to-site renormalisation group. This procedure has the advantage of drastically reducing the surface effects, giving the possibility of measuring bulk quantities with noticeable precision, and is especially useful for calculating the fractal dimension for systems which slowly converge to their asymptotic value. Moreover it provides a way of obtaining valuable information on the dynamics of growth models.

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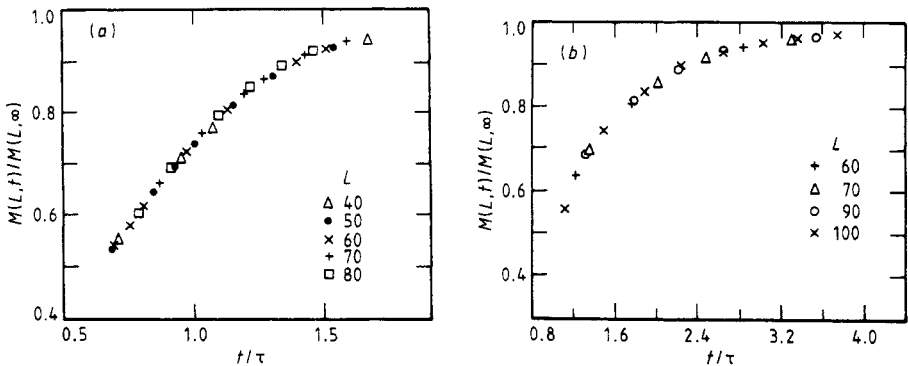


Figure 4. Data collapse plotted as  $M(L, t)/M(L, \infty)$  for various values of  $L$  for: (a) Eden model; (b) DLA model.

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